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THE LIMITATIONS OF THE PATCH TEST

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SUMMARY

A simple approximation by nonconforming finite elements is presented that passes the patch test of Irons and Strang but does not yield approximate solutions converging to the solution of the given boundary value problem. It is constructed from continuous piecewise linear functions perturbed by step functions. Further, strange convergence properties of such approximations are explained in all details because they may be typical for the behaviour of nonconforming finite elements violating the basic precondition for convergence.

INTRODUCTION

The present paper describes an approximation of the solutions of a simple one-dimensional boundary value problem by nonconforming finite elements passing the patch test of Irons and Strang[1–3] but not converging to the solution of the given boundary value problem. Thus, in contrast to the opinion of many authors, success in this test is not sufficient for convergence. We only remark in passing that neither does this test yield a necessary convergence condition[4,5] as may be seen, for instance, from finite elements satisfying the required continuity conditions approximately or excepting sufficiently small subsets of nodal points. Our example may be generalized in an obvious way to two dimensions: a quadratic domain is subdivided by quadratic elements; the associated trial spaces consist of the well-known conforming bilinear functions together with nonconforming perturbations by suitable step functions.

The basic idea of the Irons patch test is that if the mesh is fine and the patches are small the solution of the given problem over any patch may well be approximated by a linear function. If the patch test is passed, a linear function is approximated exactly so that the approximation solutions approximate the solution of the boundary value problem. With respect to this conclusion, Irons and Razzaque[1] (page 560) state the noteworthy reservation: 'provided that small perturbations from uniform conditions do not cause a disproportionate response in the patch', but continue, 'we hope to prevent this by insisting that K is positive definite' (K denotes the matrix of the approximating equations). In our example, exactly the apprehended response occurs. A linear solution $u = p_1$ of the boundary value problem is reproduced exactly by $u_h = p_1$ so that u_h is a conforming function. The nonconforming nature of the trial spaces disappears in this case. It can only make its appearance if the solution u deviates from a linear function. The nonconforming part of the trial functions consists of step functions having no continuity properties at all and thus causing essential perturbations. Hence the example shows that, contrary to the above-cited hope, the positive definiteness does not necessarily prevent the patches from a disproportionate response. Note that the introduction to the third section continues the discussion of Irons's original idea.

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Our example clearly refutes the conception of Irons and Razzaque[1] (page 560 below) that their patch test be applicable to a wide class of nonconforming elements G. Strang has conjectured that the test is valid for conventional nonconforming elements (the nodal finite elements described in Reference 3). At present this conjecture is unsettled. In view of this situation it is important to note that the papers of de Arantes e Oliveira,[5] Ciarlet[6] and Lascaux and Lesaint,[7] in fact do not use this test but prove convergence by deriving appropriate error estimates. In References 6 and 7 a so-called local patch test is used whose form, however, depends on the special nonconforming element.

Note that we have established[8,9] a new, generalized patch test which yields a both necessary and sufficient convergence condition for approximations by nonconforming elements. It is proved[9] that a series of well-known nonconforming elements pass this test and constitute convergent approximations also of boundary value problems with non-smooth coefficients.

The example is further used in the present paper to illustrate some strange properties[10] of those nonconforming elements which do not pass the generalized patch test or an equivalent convergence condition. In particular, the astonishing fact is explained that approximations by the spaces V_h converge but not to the desired solution of the given problem. This phenomenon occurs in general, for example, when sequences of subspaces are generated by successive subdivisions of a mesh. Increasing sequences of subspaces always converge, possibly however to too great a limit. The purpose of the third and fourth sections is to explain such strange properties in all details. In particular, it will be proved that the limits of convergent sequences of approximation solutions in V_h are again solutions of a variational equation. These solutions are, in general, different from the solution of the given problem. This defect cannot be remedied by modifying the inhomogeneous term or load vector, except that the solution to be approximated is known. Finally it is shown that the usual form of the inhomogeneous term leads to approximations asymptotically overestimating the strain energy of the given problem. Presumably, approximations by the subspaces W_h do not possess the above-described convergence behaviour due to the fact that the subspaces $[w_2, w_5, \dots, w_{3n-1}]$ do not approximate $L^2(I)$.

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THE BOUNDARY VALUE PROBLEM AND NONCONFORMING APPROXIMATIONS

Consider the Dirichlet boundary value problem

$$-\frac{d^2u}{dx^2}+u=f \text{ in } I, \qquad u(a)=\alpha, \qquad u(b)=\beta \tag{1}$$

for real-valued inhomogeneous terms $f \in L^2(I)$, solutions $u \in H^2(I)$, and some bounded open interval $I=(a, b)$ of the real line. In the following, g denotes the linear interpolate of the boundary values α at $x = a$ and β at $x = b$. Problem (1) is equivalent to the variational equation

$$u-g \in H_0^1(I); \qquad \int_I (D\varphi Du + \varphi u)\,dx = \int_I \varphi f\,dx, \qquad \varphi \in H_0^1(I) \tag{2}$$

where $D\varphi$ denotes the derivative $d\varphi/dx$ of functions $\varphi \in H^1(I)$. The above problem is approximated by means of subdivisions of the interval I into m equidistant open subintervals $I_j = (x_{j-1}, x_j)$ of length h. The subdivisions and the points $x_j = a + jh$ depend on m or the mesh width

$$h = h_m = \frac{b-a}{m}, \qquad m = 1, 2, \ldots$$

The variational equation (2) is approximated by a sequence of variational equations of the form

$$u_h - g \in V_h; \qquad \sum_{j=1}^{m} \int_{I_j} (\mathrm{D}\varphi_h \mathrm{D}u_h + \varphi_h u_h)\,\mathrm{d}x = \int_I \varphi_h f\,\mathrm{d}x, \qquad \varphi_h \in V_h \tag{3}$$

for all mesh widths $h = h_m$. The spaces V_h are finite dimensional and consist of piecewise linear functions. Let v_j be the well-known continuous piecewise linear roof functions specified by the property

$$v_j(x_k) = \delta_{jk}, \qquad j, k = 0, \ldots, m$$

Next let w_j be the step functions

$$w_j(x) = 1, \qquad x_{j-1} < x < x_j; \qquad w_j(x) = 0, \qquad \text{elsewhere};$$

for $j = 1, \ldots, m$. Then V_h is the linear subspace of $L^2(I)$ spanned by $v_1, \ldots, v_{m-1}$ and $w_2, \ldots, w_{m-1}$,

$$V_h = [v_1, \ldots, v_{m-1}, w_2, \ldots, w_{m-1}]$$

Every function $\varphi \in V_h$ has the representation

$$\varphi(x) = \sum_{j=1}^{m-1} y_j v_j(x) + \sum_{j=2}^{m-1} z_j w_j(x), \qquad x \in [a, b] \tag{4}$$

The first sum on the right-hand side is the conforming, continuous part $\varphi^1 \in H_0^1(I)$ and the second sum the nonconforming, discontinuous part $\varphi^0 \in L^2(I)$ of φ. Obviously, the coefficients y_j, z_j of this representation are the nodal values

$$y_j = \varphi^1(x_j), \qquad z_j = \varphi^0(x_{j-1/2}) = \frac{1}{h}\int_{I_j} \varphi^0(x)\,\mathrm{d}x \tag{5}$$

of φ^1, φ^0. Note that the trial functions $\varphi \in V_h$ jump at the mesh points x_j:

$$\varphi(x_j + 0) - \varphi(x_j - 0) = \varphi^0(x_j + 0) - \varphi^0(x_j - 0) = z_{j+1} - z_j, \qquad j = 1, \ldots, m-1, \tag{6}$$

where $z_1 = z_m = 0$.

The method of nonconforming finite elements defines the derivative $D\varphi$ of a trial function $\varphi \in V_h$ by

$$\mathrm{D}\varphi(x) = \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x), \qquad x_{j-1} < x < x_j, \qquad j = 1, \ldots, m$$

Thus the derivative of the step function φ^0 vanishes so that we have

$$\mathrm{D}\varphi^1 = \mathrm{D}\varphi, \qquad \mathrm{D}\varphi^0 = 0, \qquad \varphi = \varphi^1 + \varphi^0 \in V_h$$

The conforming part φ^1 and the nonconforming part φ^0 may, therefore, be obtained by

$$\varphi^1(x) = \int_a^x \mathrm{D}\varphi(s)\,\mathrm{d}s, \qquad \varphi^0(x) = \varphi(x) - \int_a^x \mathrm{D}\varphi(s)\,\mathrm{d}s, \qquad x \in [a, b]. \tag{7}$$

In the present context, convergence of the method of nonconforming finite elements may be viewed as convergence in the space $L^{1,2}(I)$ of all vector-valued functions $\mathbf{u}=(u, u')$ with components $u, u' \in L^2(I)$. This is a Hilbert space with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_I (u'v' + uv)\,\mathrm{d}x, \qquad \mathbf{u}, \mathbf{v} \in L^{1,2}(I)$$

The natural embedding of functions $u \in H^m(I)$ into the space $L^{1,2}(I)$ is obtained by $\mathbf{u}=(u, \mathrm{D}u)$. The associated natural embedding of the Sobolev spaces $H^m(I)$ into $L^{1,2}(I)$ will be denoted by ${}_EH^m(I)$. In a similar way, the natural embeddings ${}_EV_h$ of the trial spaces V_h of piecewise linear functions into $L^{1,2}(I)$ are defined by

$$\mathbf{u}_h = (u_h, u'_h),\ u'_h(x) = \mathrm{D}u_h(x), \qquad x \in I_j, \qquad j = 1, \ldots, m$$

Using these concepts, one has

$$\|\mathbf{u}_h - \mathbf{u}\|^2 = \sum_{j=1}^{m} \int_{I_j} (|\mathrm{D}u_h - \mathrm{D}u|^2 + |u_h - u|^2)\,\mathrm{d}x$$

Consequently, the sequence of solutions $u_h \in V_h$ of (3) converges to the solution $u \in H_0^1(G)$ of (1), (2) in the sense of the method of nonconforming finite elements if, and only if, the associated sequence of embedded functions $\mathbf{u}_h$ converges to $\mathbf{u}$ in $L^{1,2}(I)$.

On setting

$$E_0 = {}_EH_0^1(I), \qquad E_h = {}_EV_h, \qquad h = h_m, \qquad m = 1, 2, \ldots$$

the variational equations (2), (3) take on the simple form

$$\mathbf{u} - \mathbf{g} \in E_0; \qquad (\boldsymbol{\varphi}, \mathbf{u}) = l(\boldsymbol{\varphi}), \qquad \boldsymbol{\varphi} \in E_0 \tag{8a}$$

and

$$\mathbf{u}_h - \mathbf{g} \in E_h; \qquad (\boldsymbol{\varphi}_h, \mathbf{u}_h) = l(\boldsymbol{\varphi}_h), \qquad \boldsymbol{\varphi}_h \in E_h \tag{8b}$$

where $\mathbf{u}$ is the embedded solution u of (1), (2) and the right-hand side of these equations is defined by

$$l(\boldsymbol{\varphi}) = \int_I \varphi f\,\mathrm{d}x, \qquad \boldsymbol{\varphi} = (\varphi, \varphi') \in L^{1,2}(I) \tag{9}$$

SUCCESS IN THE PATCH TEST AND DIVERGENCE OF THE APPROXIMATION SOLUTIONS

Does the sequence of solutions u_h converge to the solution u for $h \to 0$? This question will be answered by studying the associated discretization error. Let P_h be the orthogonal projection of $L^{1,2}(I)$ on to the subspace $E_h = {}_EV_h$ and let $|\mathbf{v}, E_h|$ be the shortest distance from $\mathbf{v}$ to E_h. Using the solution $\mathbf{u}$ of (8a), the associated error functional d is defined by

$$d(\boldsymbol{\varphi}) = l(\boldsymbol{\varphi}) - (\boldsymbol{\varphi}, \mathbf{u}), \qquad \boldsymbol{\varphi} \in L^{1,2}(I) \tag{10}$$

Obviously, $(\boldsymbol{\varphi}_h, \mathbf{v}) = (\boldsymbol{\varphi}_h, P_h\mathbf{v})$ for all $\boldsymbol{\varphi}_h \in E_h$. By virtue of the variational equations (8b), the norm of the error functional d on E_h can be written in the form

$$\|d\|_{E_h} = \sup_{0 \neq \boldsymbol{\varphi} \in E_h} \frac{|d(\boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|} = \|\mathbf{v}_h - P_h\mathbf{v}\|$$

where $\mathbf{v}=\mathbf{u}-\mathbf{g}$, $\mathbf{v}_h=\mathbf{u}_h-\mathbf{g}$. The function $\mathbf{v}_h-P_h\mathbf{v}$ is orthogonal to the subspace E_h. Pythagoras's theorem thus yields the fundamental discretization error equation

$$\|\mathbf{u}_h-\mathbf{u}\|^2=|\mathbf{u}-\mathbf{g},E_h|^2+\|d\|^2_{E_h'} \tag{11}$$

for the solutions $\mathbf{u}_h$ of the approximating variational equations (8b).

The solution u of (1), (2) belongs to $H^2(I)\cap H_0^1(I)$ and can, therefore, be approximated by piecewise linear functions. Using the formulae (4), (5), (7) for $\varphi=u-g$, one obtains the coefficients y_j, z_j of a piecewise linear interpolate $\varphi_h^I\in V_h$. As $u-g$ is continuous, we have $\varphi^1=\varphi$, $\varphi^0=0$ so that $y_j=\varphi(x_j)$, $z_j=0$. Consequently,

$$\varphi_h^I(x)=\sum_{j=1}^{m-1}\varphi(x_j)v_j(x),\qquad x\in[a,b]$$

is the continuous piecewise linear interpolate of $u-g$ at the mesh points x_j. The well-known associated error estimate thus yields the approximability condition

$$|\mathbf{u}-\mathbf{g},E_h|\leqslant\|\mathbf{u}-\mathbf{g}-\boldsymbol{\varphi}_h^I\|\leqslant\gamma h|u|_2 \tag{12}$$

uniformly for all $h=h_m$, $m=1,2,\ldots$.

The Irons patch test considers the case that the solution u is a linear function p_1. Such a function solves the boundary value problem (1) for $f=p_1$, $\alpha=p_1(a)$, $\beta=p_1(b)$. Correspondingly, we have $g=p_1$. Then the test is to see whether the associated nonconforming approximations u_h are identical with p_1. In view of (11), the error of these approximations is

$$\|\mathbf{u}_h-\mathbf{p}_1\|=\|d\|_{E_h'}$$

because $u-g=p_1-p_1=0$ and $|0,E_h|=0$. Hence the two statements

$$u_h=p_1 \tag{13a}$$

and

$$d(\boldsymbol{\varphi}_h;\mathbf{p}_1)=0,\qquad \boldsymbol{\varphi}_h\in E_h \tag{13b}$$

are equivalent. Evidently, it suffices to verify the second statement for all basis functions in E_h. Since the conforming basis functions $\mathbf{v}_j\in E_h\cap E_0$ give $d(\mathbf{v}_j;\mathbf{u})=0$, as is readily seen from (8a), (10), the two statements above are further equivalent to

$$d(\mathbf{w}_j;\mathbf{p}_1)=0 \tag{13c}$$

for all nonconforming basis functions $\mathbf{w}_j$ in E_h.

This is the patch test of Strang[2] and Strang and Fix[3] (page 176) in our notation. Note that the formulation of the patch test by Brown[11] (page 75) and Ciarlet[12] (page 223) is the same as (13b).

By partial integration the error functional d becomes

$$\begin{aligned}d(\boldsymbol{\varphi}_h;\mathbf{u})&=\sum_{j=1}^{m}\left\{\int_{I_j}\varphi_h f\,\mathrm{d}x-\int_{I_j}(\mathrm{D}\varphi_h\mathrm{D}u+\varphi_h u)\,\mathrm{d}x\right\}\\&=\sum_{j=1}^{m-1}\{\varphi_h(x_j+0)-\varphi_h(x_j-0)\}\mathrm{D}u(x_j)\\&=\sum_{j=2}^{m-1}z_j\{\mathrm{D}u(x_{j-1})-\mathrm{D}u(x_j)\},\end{aligned} \tag{14}$$

because u satisfies the differential equation (1), the trial functions φ_h vanish at $x = a$ and $x = b$, and make the jumps (6) at x_j. The nonconforming basis functions thus yield

$$d(\mathbf{w}_j; \mathbf{u}) = \mathrm{D}u(x_{j-1}) - \mathrm{D}u(x_j), \qquad j = 2, \ldots, m-1$$

Every polynomial $u = p_1$ of first degree has a constant derivative $\mathrm{D}u$ so that

$$d(\mathbf{w}_j; \mathbf{p}_1) = 0, \qquad j = 2, \ldots, m-1$$

and, consequently, the patch test is passed.

But, the sequence of solutions u_h of (3) does not converge to the solutions u of the given problem (1), (2), as will now be shown. From (14) it follows that

$$d(\varphi_h; \mathbf{u}) = -h \sum_{j=2}^{m-1} z_j u''(I_j), \qquad \varphi_h \in E_h$$

using the mean values

$$u''(I_j) = \frac{1}{h} \int_{I_j} \mathrm{D}^2 u \, \mathrm{d}x, \qquad j = 1, \ldots, m$$

The coefficients z_j are uniquely determined by the representation (4) of φ_h. Choosing the special sequence of step functions

$$\varphi_h(x) = \sum_{j=2}^{m-1} v(x_j) w_j(x), \qquad x \in [a, b]$$

where $v = u - g$, it follows that

$$d(\varphi_h; \mathbf{u}) = -h \sum_{j=2}^{m-1} v(x_j) v''(I_j)$$

since $v'' = \mathrm{D}^2 v = \mathrm{D}^2 u$. Obviously,

$$d(\varphi_h; \mathbf{u}) \to -\int_I v \mathrm{D}^2 v \, \mathrm{d}x = \int_I |\mathrm{D}v|^2 \, \mathrm{d}x = \|\mathrm{D}v\|_0^2 \; (h \to 0)$$

It is well known that the sequence (φ_h) converges to v in $L^2(I)$ for $h \to 0$. The derivative φ_h' of φ_h, in the sense of nonconforming finite elements, is equal to zero. Hence

$$\varphi_h = (\varphi_h, \varphi_h') \to (v, 0) \quad \text{in} \quad L^{1,2}(I)$$

and so

$$\|\varphi_h\|^2 = \int_I (|\varphi_h'|^2 + |\varphi_h|^2) \, \mathrm{d}x \to \int_I |v|^2 \, \mathrm{d}x = \|v\|_0^2 \; (h \leftarrow 0)$$

Consequently, one has

$$\|d\|_{E_h'} \geq \frac{|d(\varphi_h; \mathbf{u})|}{\|\varphi_h\|_0} \to \frac{\|\mathrm{D}v\|_0^2}{\|v\|_0} (h \to 0)$$

The discretization error equation (11) then implies the statement

$$\liminf_{h \to 0} \|\mathbf{u}_h - \mathbf{u}\| = \liminf_{h \to 0} \|d\|_{E_h'} \geq \frac{\|\mathrm{D}v\|_0^2}{\|v\|_0} > 0$$

for each solution $u \in H^2(I) \cap H_0^1(I)$ such that $v = u - g \neq 0$. Thence it is seen that u_h cannot converge to u in general.

The generalized patch test[8,9] requires here

$$\lim_{h \to 0} \sum_{j=1}^{m-1} \psi(x_j)(\varphi_h(x_j+0) - \varphi_h(x_j-0)) = 0$$

for all test functions $\psi \in C_0^\infty(\mathbb{R}^n)$ and every bounded sequence of functions $\varphi_h \in V_h$. This condition is not valid in the present example as is proved in just the same way as above.

A DIVERGENT FINITE ELEMENT PASSING THE TEST

This section deals with a sequence of subspaces $W_h \subset V_h$ defined by finite elements in the sense of the general definition by Ciarlet.[12] Due to the discontinuities of the trial functions, the degrees-of-freedom have the form of left- and right-sided limits of function values and of jumps at discontinuities.

The idea of Irons's patch test has already been described in the Introduction. Consider now fine subdivisions of the interval I. For the spaces W_h each subinterval of the form $[x_{3k}, x_{3l}]$ for $k < l$ is a patch of elements. To any solution u of (1) choose a small patch length such that u can be approximated within a prescribed accuracy by a linear function in each patch of this length. The interval I may, obviously, be covered by overlapping patches. For all sufficiently small mesh widths h our element passes the test: a state of constant strain on a patch is exactly reproduced by the nonconforming approximation. Irons and Razzaque[1] (page 559) claim the following: if two overlapping patches can reproduce any given state of constant strain they should combine into a larger successful patch. If this were true the whole interval I would be successful, i.e. would yield convergent approximations. As we shall see, however, the solutions u_h of the approximating equations for W_h do not converge to the solution u of (1). In this way, a contradiction to the above hypothesis of Irons and Razzaque is established.

The finite elements in this section are closed intervals $K = [x_0, x_3]$ of length $3h$ and the associated space of real functions is the span

$$M = [v_0, v_1, v_2, v_3, w_2]$$

Functions in this space have the form

$$\chi(x) = \sum_{j=0}^{3} y_j v_j(x) + z_2 w_2(x), \qquad x_0 \leq x \leq x_3$$

(Figure 1). Obviously, the coefficients of the representation are the degrees-of-freedom

$$y_0 = \chi(x_0+0), \qquad y_1 = \chi(x_1-0), \qquad y_2 = \chi(x_2+0), \qquad y_3 = \chi(x_3-0),$$
$$z_2 = \chi(x_1+0) - \chi(x_1-0) = -(\chi(x_2+0) - \chi(x_2-0))$$

of the function χ. The right-hand sides of these equations specify linearly independent linear forms over M. They are defined as well over the space of continuous functions on $[x_0, x_3]$. The functions $v_0, \ldots, v_3, w_2$ constitute a basis of M, biorthogonal to these linear forms. Thus we have a finite element in the sense of the general definition by Ciarlet[12] (page 78).

This element yields nonconforming approximations of the type considered above for our boundary value problem. The approximations are specified on the whole interval I by the subspaces

$$W_h = [v_1, \ldots, v_{3n-1}, w_2, w_5, \ldots, w_{3n-1}] \subset V_h$$

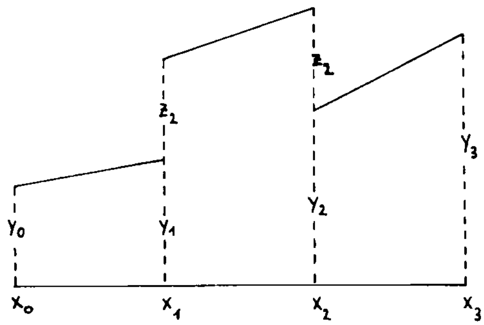

Figure 1. The nonconforming finite element passing the patch test

for all mesh widths $h = h_{3n}$, $n = 1, 2, \ldots$. The trial functions in W_h have the form

$$\psi(x) = \sum_{j=1}^{3n-1} y_j v_j(x) + \sum_{k=1}^{n} z_{3k-1} w_{3k-1}(x), \qquad x \in [a, b] \tag{15}$$

Note that these functions satisfy the homogeneous boundary conditions $\psi(a) = \psi(b) = 0$. Now y_j, z_{3k-1} are the degrees-of-freedom

$$y_{3k} = \psi(x_{3k}+0) = \psi(x_{3k}-0), \qquad k = 1, \ldots, n-1$$

$$y_{3k-2} = \psi(x_{3k-2}-0), \qquad y_{3k-1} = \psi(x_{3k-1}+0)$$

$$z_{3k-1} = \psi(x_{3k-2}+0) - \psi(x_{3k-2}-0) = -(\psi(x_{3k-1}+0) - \psi(x_{3k-1}+0)) \tag{16}$$

for $k = 1, \ldots, n$.

The approximating equations for this class of nonconforming elements again have the form (3) for $3n$ and W_h, instead of m and V_h, or the form (8) where $E_0 = {}_E H_0^1(I)$ and $E_h = {}_E W_h$ for $h = h_{3n}$, $n = 1, 2, \ldots$. The associated discretization error equation is found in (11). The approximability condition

$$|\mathbf{u} - \mathbf{g}, {}_E W_h| \to 0 \ (h \to 0)$$

is proved just as in the previous section. Using (14), the error functional d on the subspaces W_h reads

$$d(\boldsymbol{\psi}_h; \mathbf{u}) = \sum_{k=1}^{n} z_{3k-1}\{\mathrm{D}u(x_{3k-2}) - \mathrm{D}u(x_{3k-1})\}, \qquad \psi_h \in W_h \tag{17}$$

Hence the nonconforming basis functions w_{3k-1} in W_h pass the patch test (13),

$$d(\mathbf{w}_{3k-1}; \mathbf{p}_1) = 0, \qquad k = 1, \ldots, n$$

However, the approximation solutions in W_h do not converge to the solution u of (1), either. To see this, put $v = u - g$ and choose the special sequence of trial functions

$$\psi_h(x) = \sum_{k=1}^{n} v(x_{3k-3/2}) w_{3k-1}(x)$$

By virtue of (17), we now have

$$d(\boldsymbol{\psi}_h; \mathbf{u}) = -h \sum_{k=1}^{n} v(x_{3k-3/2}) v''(I_{3k-1})$$

since $v'' = \mathrm{D}^2 v = \mathrm{D}^2 u$. For brevity, denote vv'' by w. Then

$$h \sum_{k=1}^{n} |v(x_{3k-3/2})v''(I_{3k-1}) - w(I_{3k-1})| \leq h\|\mathrm{D}v\|_0\|\mathrm{D}^2 v\|_0 \to 0$$

and

$$h \sum_{k=1}^{n} |w(I_{3k}) - w(I_{3k-1})| \leq \int_{a+h}^{b} |w(x) - w(x-h)|\,\mathrm{d}x \to 0$$

$$h \sum_{k=1}^{n} |w(I_{3k-1}) - w(I_{3k-2})| \leq \int_{a}^{b-h} |w(x+h) - w(x)|\,\mathrm{d}x \to 0$$

for $h \to 0$. This implies the convergence of

$$3d(\boldsymbol{\psi}_h; \mathbf{u}) \to -\int_I w\,\mathrm{d}x = \int_I |\mathrm{D}v|^2\,\mathrm{d}x = \|\mathrm{D}v\|_0^2 \quad (h \to 0)$$

Further

$$3\int_I |\psi_h(x)|^2\,\mathrm{d}x = 3h \sum_{k=1}^{n} |v(x_{3k-3/2})|^2 \to \int |v|^2\,\mathrm{d}x = \|v\|_0^2$$

for $h \to 0$, because the sums are composite midpoint rules to the mesh widths $3h$ approximating the integral of $|v|^2$ over the interval $[a, b]$. As $\psi_h' = \mathrm{D}\psi_h = 0$ it follows that

$$\|\boldsymbol{\psi}_h\|^2 = \int_I (|\psi_h'|^2 + |\psi_h|^2)\,\mathrm{d}x \to \tfrac{1}{3}\|v\|^2 \quad (h \to 0)$$

and thus

$$\|d\|_{E_h'} \geq \frac{|d(\boldsymbol{\psi}_h; \mathbf{u})|}{\|\boldsymbol{\psi}_h\|} \to \frac{\|\mathrm{D}v\|_0^2}{\sqrt{3}\|v\|_0} > 0 \quad (h \to 0)$$

provided that $v = u - g \neq 0$. By virtue of the error equation (11), this relation shows the divergence of the nonconforming approximations in W_h.

STRANGE PROPERTIES

The example exhibits some strange properties that may occur in approximations by nonconforming finite elements which do not pass the generalized patch test or some equivalent convergence condition. From now on we consider only homogeneous boundary conditions in (1), i.e. $\alpha = \beta = 0$ and thus $g = 0$ in the generalized boundary value problems (2), (3) and (8). First, let us again regard the approximating variational equations (3) or (8). The spaces $E_h = {}_E V_h$ are not contained in $E_0 = {}_E H_0^1(I)$. Therefore, strictly speaking, the inhomogeneous term of the given variational equation has first to be expanded in the form (9) on to all of $L^{1,2}(I)$ and then, in equations (3) or (8), to be restricted to the subspaces E_h. This process is, however, not unique. For, the general form of continuous linear forms on $L^{1,2}(I)$ reads

$$l(\boldsymbol{\varphi}) = \int_I (\varphi' f_1 + \varphi f_0)\,\mathrm{d}x, \qquad \boldsymbol{\varphi} = (\varphi, \varphi') \in L^{1,2}(I). \tag{18}$$

As one easily sees, this linear form possesses the representation

$$l(\boldsymbol{\varphi}) = \int_I \varphi f\,\mathrm{d}x, \qquad \boldsymbol{\varphi} \in {}_E H_0^1(I),$$

if, and only if, the condition

$$f_1 \in H^1(I), \ -Df_1 + f_0 = f, \tag{19}$$

holds. In the following we shall use the extended form (18) as inhomogeneous term in the variational equation (8). The original form can always be regained by choosing $f_0 = f$, $f_1 = 0$.

A particularly remarkable property of nonconforming approximations is the fact that the approximation solutions may converge but not to the desired solution of the given problem. More precisely, we shall show that the solutions $\mathbf{u}_h \in E_h$ of the variational equations under consideration,

$$\sum_{j=1}^{m} \int_{I_j} (D\varphi Du_h + \varphi u_h)\,dx = \sum_{j=1}^{m} (D\varphi f_1 + \varphi f_0)\,dx, \qquad \varphi \in V_h, \tag{20}$$

converge to the solution $\mathbf{z} = (z, z')$ of the variational equation

$$\mathbf{z} \in L; \quad \int_I (\varphi' z' + \varphi z)\,dx = \int_I (\varphi' f_1 + \varphi f_0)\,dx, \qquad \boldsymbol{\varphi} = (\varphi, \varphi') \in L \tag{21}$$

for all $f_0, f_1 \in L^2(I)$. The subspace $L \subset L^{1,2}(I)$ is defined by

$$L = {}_E H_0^1(I) + L^2(I)x[0] \tag{22}$$

where by $L^2(I)x[0]$ is meant the subspace of all functions $\mathbf{w} \in L^{1,2}(I)$ of the form $\mathbf{w} = (w, 0)$ for $w \in L^2(I)$. Every element in L has the form

$$\mathbf{v} + \mathbf{w}, \ \mathbf{v} = (v, Dv) \in {}_E H_0^1(I), \ \mathbf{w} = (w, 0) \in L^2(I)x[0]$$

Problem (21) has a unique solution $\mathbf{z}$ in L that will be determined explictly in the next section.

Theorem. The subspaces E_h are contained in L and $\mathbf{u}_h$ is the orthogonal projection of $\mathbf{z}$ on to $E_h = {}_E V_h$ in $L^{1,2}(I)$. The sequence of solutions $\mathbf{u}_h = (u_h, u_h')$ of the variational equations (20) *converges to the solution $\mathbf{z} = (z, z')$ of the variational equation* (21) *for all inhomogeneous terms with arbitrary $f_0, f_1 \in L^2(I)$.*

Proof: (i) Every trial function $\varphi_h \in V_h$ has the representation $\varphi_h = \varphi_h^1 + \varphi_h^0$, where φ_h^1 is the conforming and φ_h^0 the nonconforming part of φ_h, as defined in (7). The embedded trial functions $\boldsymbol{\varphi}_h \in E_h = {}_E V_h$ thus have the form $\boldsymbol{\varphi}_h = \boldsymbol{\varphi}_h^1 + \boldsymbol{\varphi}_h^0$. Evidently, $\boldsymbol{\varphi}_h^1 \in {}_E H_0^1(I)$ and $\boldsymbol{\varphi}_h^0 \in L^2(I)x[0]$ because $D\varphi_h^0 = 0$. Consequently, the subspaces E_h are contained in L. From (20), (21) it then follows that $\mathbf{u}_h$ is the orthogonal projection of $\mathbf{z}$ on to the subspace E_h of $L^{1,2}(I)$ and so

$$\|\mathbf{z} - \mathbf{u}_h\| = |\mathbf{z}, E_h| = \min_{\boldsymbol{\varphi}_h \in E_h} \|\mathbf{z} - \boldsymbol{\varphi}_h\|$$

(ii) Now $\mathbf{z} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in {}_E H_0^1(I)$, $\mathbf{w} \in L^2(I)x[0]$ or $v \in H_0^1(I)$, $w \in L^2(I)$. The conforming piecewise linear functions φ_h^1 in V_h approximate $H_0^1(I)$ and the nonconforming step functions φ_h^0 in V_h approximate $L^2(I)$. Hence there exist such sequences of functions with the property

$$\varphi_h^1 \to v \text{ in } H_0^1(I), \qquad \varphi_h^0 \to w \text{ in } L^2(I)(h \to 0)$$

The embedded sequences $\boldsymbol{\varphi}_h^1 = (\varphi_h^1, D\varphi_h^1)$, $\boldsymbol{\varphi}_h^0 = (\varphi_h^0, 0)$ then converge to $\mathbf{v}$, $\mathbf{w}$ in $L^{1,2}(I)$ for $h \to 0$. Using the above error equation, it thus follows that

$$\|\mathbf{z} - \mathbf{u}_h\| \leq \|\mathbf{v} + \mathbf{w} - (\boldsymbol{\varphi}_h^1 + \boldsymbol{\varphi}_h^0)\| \leq \|v - \varphi_h^1\|_1 + \|w - \varphi_h^0\|_0 \to 0$$

for $h \to 0$. □

CONVERGENCE BEHAVIOUR OF APPROXIMATING STRAIN ENERGIES

It has been shown in the preceding section that the solutions $\mathbf{u}_h$ of the approximating variational equations (20) converge to the solution $\mathbf{z}$ of (21). Correspondingly, the energy approximations $\|\mathbf{u}_h\|^2$ converge to $\|\mathbf{z}\|^2$, i.e.

$$\int_I (|u_h'|^2+|u_h|^2)\,\mathrm{d}x \to \int_I (|z'|^2+|z|^2)\,\mathrm{d}x \quad (h\to 0)$$

It is possible, in this example, to establish an explicit representation of $\mathbf{z}$ and its norm $\|\mathbf{z}\|$ by the given data f_0, f_1. For brevity, we write $f_1(I)$ for the mean value of f_1 over the interval I. The solution $\mathbf{z}$ of (21) has the form

$$\mathbf{z}=(f_0, f_1-f_1(I)) \tag{23}$$

and the associated norm has the representation

$$\|\mathbf{z}\|^2=\int_I (|f_0|^2+|f_1|^2)\,\mathrm{d}x-(b-a)|f_1(I)|^2 \tag{24}$$

Obviously, this function $\mathbf{z}$ is a solution of the variational equation (21) if $\int_I \varphi'\,\mathrm{d}x=0$ for all $\boldsymbol{\varphi}=(\varphi,\varphi')\in L$. Now, by definition of L, we have $\boldsymbol{\varphi}=\boldsymbol{\chi}+\boldsymbol{\psi}$, where $\boldsymbol{\chi}\in {}_EH_0^1(I)$, $\boldsymbol{\psi}\in L^2(I)x[0]$. So $\varphi'=\mathrm{D}\chi$ for some function $\chi\in H_0^1(I)$ and hence

$$\int_I \varphi'\,\mathrm{d}x=\chi(b)-\chi(a)=0$$

It has further to be verified that $\mathbf{z}$ belongs to L.Choose

$$v(x)=\int_a^x (f_1(t)-f_1(I))\,\mathrm{d}t, \qquad v'(x)=f_1(x)-f_1(I)$$

$$w(x)=f_0(x)-v(x), \qquad w'(x)=0, \qquad x\in[a,b]$$

Then $v\in H_0^1(I)$ and

$$\mathbf{v}=(v,v')\in {}_EH_0^1(I), \qquad \mathbf{w}=(w,0)\in L^2(I)x[0]$$

Consequently,

$$\mathbf{z}=(f_0, f_1-f_1(I))=(v+w, v')=\mathbf{v}+\mathbf{w}.$$

From the above representation it is seen that the solution $\mathbf{z}$ is equal to the embedded solution $\mathbf{u}=(u,\mathrm{D}u)$ of (1), (2) for $g=0$ if, and only if,

$$f_0=u, f_1=\mathrm{D}u+f_1(I) \tag{25}$$

In this case, condition (19) is fulfilled because u is a solution of the differential equation $-\mathrm{D}^2u+u=f$. By partial integration and summation, the inhomogeneous term in the approximating variational equations (20) becomes

$$l(\boldsymbol{\varphi})=\int_I \varphi f\,\mathrm{d}x-\sum_{j=1}^{m-1}(\mathrm{D}u)(x_j)(\varphi(x_j+0)-\varphi(x_j-0)), \qquad \varphi\in V_h$$

By virtue of (6), the sum over the constant term $f_1(I)$ is equal to zero. Thus there exists a correction term for the right-hand side $\int \varphi f\,\mathrm{d}x$ guaranteeing the convergence of the sequence

$(\mathbf{u}_h)$ to the solution $\mathbf{u}$. However, this correction makes use of the solution u and so is not available in practice.

Usually, the inhomogeneous terms of the variational equations (20), (21) are defined by

$$f_0 = f, f_1 = 0 \tag{26}$$

More generally, also f_0, f_1 are admissible which satisfy condition (19). The space ${}_EH_0^1(I)$ is a subspace of the space $L = {}_EH_0^1(I) + L^2(I)x[0]$ containing the limit $\mathbf{z}$. From the variational equations (2), for $g = 0$, and (21) it follows that the difference $\mathbf{z} - \mathbf{u}$ is orthogonal to the subspace ${}_EH_0^1(I)$ in $L^{1,2}(I)$. Hence $\mathbf{u}$ is the orthogonal projection of $\mathbf{z}$ into ${}_EH_0^1(I)$ and

$$\|\mathbf{z}\|^2 - \|\mathbf{u}\|^2 = \|\mathbf{z} - \mathbf{u}\|^2 = |\mathbf{z}, {}_EH_0^1(I)|^2 \tag{27}$$

Using this equation, one finally obtains the following statement concerning the asymptotic behaviour of the approximations $\|\mathbf{u}_h\|^2$ to the strain energy $\|\mathbf{u}\|^2$:

$$\lim_{h \to 0} \|\mathbf{u}_h\|^2 = \|\mathbf{z}\|^2 > \|\mathbf{u}\|^2$$

for all f_0, f_1 satisfying condition (19) but excepting the special case (25). This shows that the approximating strain energies $\|\mathbf{u}_h\|^2$ asymptotically overestimate the strain energy $\|\mathbf{u}\|^2$ of the solution u of the given problem (1).

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